"In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institution shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

"

BOUNDS ON INFLUENCE COEFFICIENTS FOR HEMISPHERICAL SHELLS WITH APICAL HOLES

A THESIS

Presented to

the Faculty of the Graduate Division

by

Thomas Richard Morel

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Applied Mathematics

Georgia Institute of Technology

June 1958

BOUNDS ON INFLUENCE COEFFICIENTS

FOR HEMISPHERICAL SHELLS WITH APICAL HOLES

Approved:

J. W. Wray

J. P. Vidosic

M. B. Sledd

Date Approved by Chairman: June 2, 1958

ACKNOWLEDGMENT

The author wishes to express his appreciation to Professor Marvin B. Sledd for the suggestion of the topic for this thesis and for the patient guidance which was responsible for the successful completion of the investigation.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SUMMARY

This thesis presents a method of obtaining upper and lower bounds for the direct and inverse influence coefficients of a thin hemispherical shell of constant thickness with a hole at the apex. The shell considered is stressed by a uniformly distributed shearing force H_o and bending moment M_o applied to the equatorial edge. The upper edge, surrounding the hole at the apex, is free. The loaded edge of the stressed shell undergoes a horizontal displacement u_o and a rotation β_o. The relations between the applied stresses and the resulting displacements are assumed linear, and these relations define the influence coefficients, as follows.

$$u_o = - c_{uH} H_o + c_{uM} M_o$$

$$\beta_o = c_{\beta H} H_o - c_{\beta M} M_o$$

$$H_o = - k_{uH} u_o - k_{\beta H} \beta_o$$

$$M_o = - k_{uM} u_o - k_{\beta M} \beta_o$$

The coefficients c are called the "direct influence coefficients;" the coefficients k are called the "inverse influence coefficients." An exact determination of these

influence coefficients for the shell under consideration has not yet been accomplished because of the difficulty of solving the relevant differential equations.

The derivation of the bounds on the influence coefficients is based on the minimum principles of the theory of elasticity. Suitable boundary-value problems are formulated and bounds are obtained for certain quadratic forms. These bounds are integral functionals whose argument functions (the comparison functions of the pertinent variational problem) represent stresses and displacements of the distorted hemispherical shell. A method is then presented for obtaining upper and lower bounds on the influence coefficients from the bounds previously obtained for the quadratic forms. By a suitable selection of the functions representing the varied quantities, this method makes possible the calculation of numerical bounds for each influence coefficient of a hemispherical shell with known physical characteristics.

The above techniques are illustrated by a numerical example involving a hemispherical shell with a sixty-degree apical hole. Upper and lower bounds are calculated for the two direct influence coefficients, c_{uH} and $c_{\beta M}$. The bounds obtained indicate that the influence coefficients of the particular hemispherical shell considered are approximately equal to the known influence coefficients of the semi-infinite cylindrical shell of the same radius and wall thickness.

The techniques described offer a method of determining bounds on the influence coefficients of a hemispherical shell of constant thickness with an apical hole of arbitrary size. The bounds may be determined to any desired degree of accuracy. However, the labor required to obtain them is considerable; and further study could profitably be devoted to reduction of the numerical complexities involved. It would also be useful to confirm or disprove the conjecture that the influence coefficients of a thin hemispherical shell with an apical hole of reasonable size (say, one hundred twenty degrees or less) are closely approximated in all cases by the influence coefficients of the semi-infinite cylindrical shell of the same radius.

CHAPTER I

INTRODUCTION

This thesis presents the extension to hemispherical shells of a technique previously used by Reissner and Sledd [1,5] to determine bounds on the influence coefficients of circular cylindrical shells. Whereas the original presentation dealt with problems of thin circular cylindrical shells of variable wall thickness, we are here concerned with the study of a thin hemispherical shell of constant thickness with a circular opening at the apex.

The hemispherical shell considered is stressed by the application to the equatorial edge of a uniformly distributed shearing force H_o and bending moment M_o. The edge which surrounds the opening at the apex is free. Under the influence of the force and the moment, the shell suffers a deformation. In particular, the edge along which H_o and M_o are applied undergoes a horizontal displacement u_o and a rotation β_o (Fig. 1). We assume that the stresses H_o and M_o are linearly related to the resulting deflections u_o and β_o. Thus

$$u_o = - c_{uH} H_o + c_{uM} M_o \ , \tag{1}$$

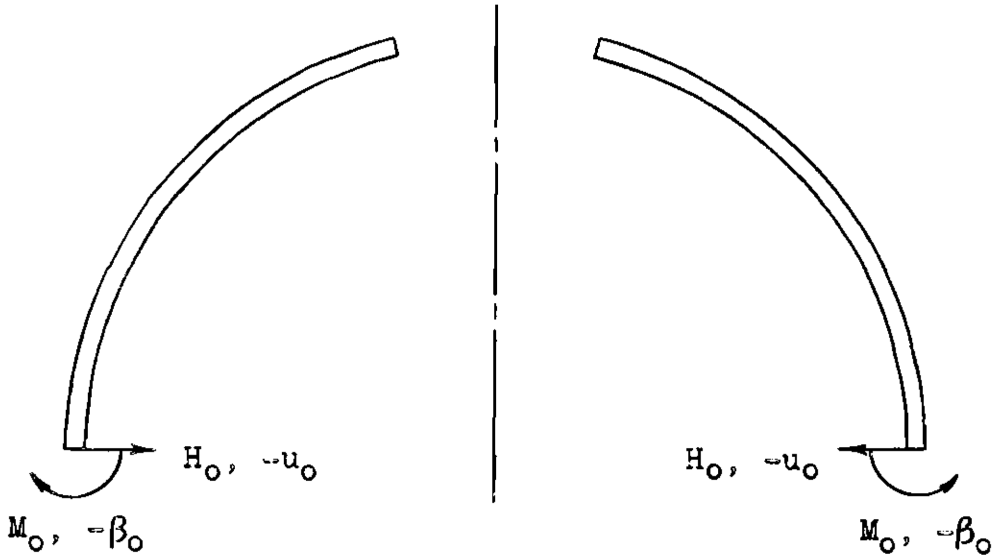

Figure 1. Cross-Section of Shell with Equatorial Edge Loading.

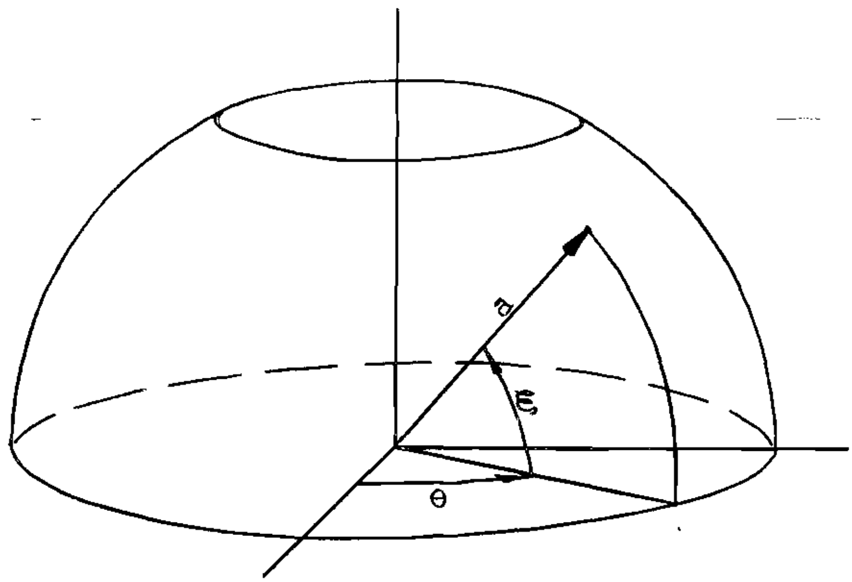

Figure 2. Middle Surface of Shell.

$$\beta_o = c_{\beta H} H_o - c_{\beta M} M_o ,$$

or

$$H_o = - k_{uH} u_o - k_{\beta H} \beta_o , \qquad (2)$$

$$M_o = - k_{uM} u_o - k_{\beta M} \beta_o .$$

The coefficients c are called the direct influence coefficients and the coefficients k are called the inverse influence coefficients. The choice of signs is such that c and k are positive.

An exact determination of the influence coefficients for the hemispherical shell has not yet been accomplished because of the difficulty of integrating the relevant differential equations. Here we present a method which will provide upper and lower bounds for these coefficients.

In Chapter II there is a brief summary of the basic physical and geometrical relationships involved in the study of thin hemispherical shells of constant thickness. Chapter III contains statements of the principles of minimum complementary energy and minimum potential energy. The principles are proven for the particular cases which are considered. Application of these minimum principles of elasticity provides four inequalities from which we extract upper and lower bounds for the four direct influence coefficients and the four inverse influence coefficients. Thus, in Chapter

IV, we exhibit sixteen bounds, each of which assumes the form of an integral functional. Eight of these functionals possess argument functions representing "varied" states of stress of the hemispherical shell. The remaining eight functionals have argument functions representing "varied" states of the displacements of the distorted hemispherical shell. These "varied" stresses and displacements are the admissible comparison functions of the pertinent variational problems. The actual stresses and displacements of the shell, although unknown, are included in the domain of admissible comparison functions, and represent the extremal argument functions. If introduced in the several integral functionals, the actual stresses and displacements would yield the exact values of the influence coefficients, the upper bound for each coefficient coinciding with the lower bound. However, expressions for the actual stresses and displacements are not available, and the problem becomes the following: to find an argument function for each integral functional such that this argument function lies in the domain of admissible comparison functions and also approximates sufficiently well the extremal argument of the pertinent variational problems. When such argument functions are found and introduced into the integral functionals, the value of each integral functional is relatively close to the exact value of the influence coefficient it bounds. Then the average value of the upper bound and the lower bound for an influence coefficient is approximately the true value, and the maximum error does not exceed half the difference between the bounds.

Chapter V presents a numerical example based on a hemispherical shell with certain specified elastic and geometric properties. Upper and lower bounds are calculated for two of the direct influence coefficients, c_{uH} and $c_{\beta M}$. The choice of the argument functions in the numerical example was facilitated by two observations: (1) that a hemispherical shell geometrically approximates a circular cylindrical shell in the vicinity of its equatorial edge and (2) that the known characteristics of the semi-infinite circular cylindrical shell indicate that forces and moments applied to an edge will produce appreciable deformations only in the region near that edge. Thus, it was assumed that the known expressions for the displacements of the stressed semi-infinite circular cylindrical shell would be a reasonable approximation of the displacements of the thin hemispherical shell. Introduction of these expressions into the appropriate integral functionals produced the bounds shown below, where each bound is expressed as the product of a numeric by the corresponding influence coefficient of the semi-infinite cylindrical shell:

$$0.950\ (a^3/2D\lambda^3) \le c_{uH} \le 1.016\ (a^3/2D\lambda^3)\ , \qquad (3)$$

$$0.983\ (a/D\lambda) \leq c_{\beta M} \leq 1.000\ (a/D\lambda).$$

From these results it is evident that the influence coefficients c_{uH} and $c_{\beta M}$ of the particular hemispherical shell considered can be approximated by 0.983 of the corresponding influence coefficients of the semi-infinite cylindrical shell and that the error in each approximation will be less than 3.5 per cent.

CHAPTER II

PHYSICS OF THE PROBLEM

In this chapter the physical quantities of the problem to be considered are defined. The basic equations relating these quantities in the case of the hemispherical shell of constant thickness are then summarized. These equations are a special case of equations previously derived by Reissner [2] for the general shell of revolution.

1. Definitions and notation.--A point on the undistorted middle surface of a hemispherical shell can be located by specifying the radius a of the middle surface, the polar angle θ , and the latitude ξ (Fig. 2). The thickness h of the shell is constant.

The state of stress of the shell is described by specifying the stress resultants N_ξ , N_θ , H and the stress couples M_ξ and M_θ (Fig. 3). The stressed middle surface of the shell undergoes a deformation, specified by the horizontal deflection u and by the rotation β of the meridian. These deflections, u and β , give rise to a circumferential strain ϵ_θ , a meridional strain ϵ_ξ , a change in circumferential curvature K_θ , and a change in meridional curvature K_ξ .

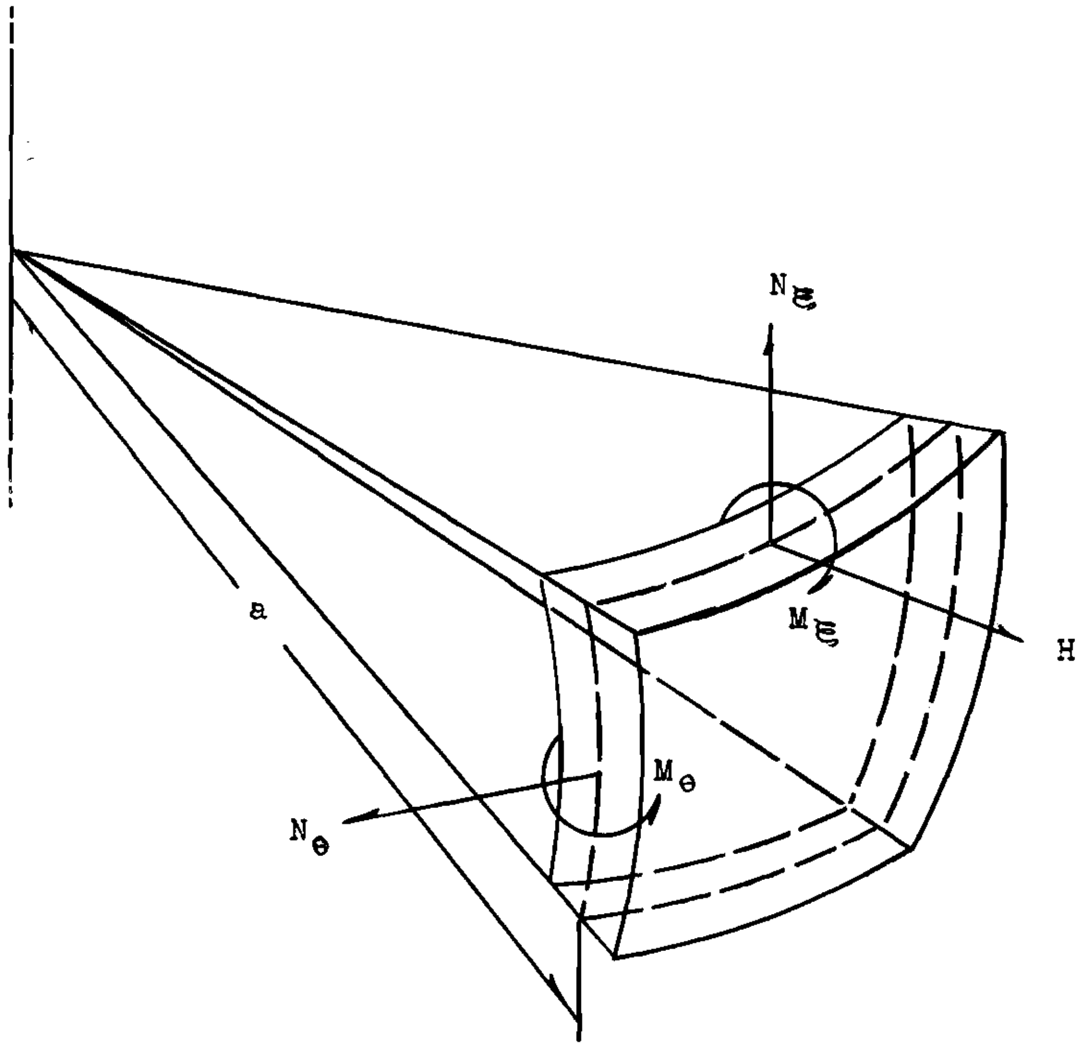

Figure 3. Stress Resultants and Stress Couples Acting on an Element of the Shell.

The elastic properties of the material composing the shell enter into the basic equations through the following quantities:

$$C = Eh \ , \tag{4}$$

$$D = Eh^3/12(1 - \nu^2) \ ,$$

$$\lambda^2 = \sqrt{3(1 - \nu^2)} \ (a/h) \ .$$

In these relationships E and ν are Young's modulus and Poisson's ratio of the material.

2. Basic equations.--Since there are no distributed surface loads on the hemispherical shell, the stress resultants and couples must satisfy the three equilibrium equations:

$$N_\xi = - H \sin\xi \ , \tag{5}$$

$$N_\theta = (H \cos\xi)' \ , \tag{6}$$

$$(M_\xi \cos\xi)' + M_\theta \sin\xi + aH \cos^2\xi = 0 \ , \tag{7}$$

where a prime (') in these equations and later ones indicates differentiation with respect to ξ .

We assume that the deformations due to transverse shear and normal stresses are negligible, and the following stress-strain relations then apply.

$$\epsilon_\xi = (N_\xi - \nu N_\theta)/C \tag{8}$$

$$\epsilon_\theta = (N_\theta - \nu N_\xi)/C \tag{9}$$

$$K_\xi = (M_\xi - \nu M_\theta)/D(1 - \nu^2) \tag{10}$$

$$K_\theta = (M_\theta - \nu M_\xi)/D(1 - \nu^2) \tag{11}$$

If the strains are expressed in terms of the displacements u and β, then four additional relationships result.

$$(a \sin\xi)\epsilon_\xi = -u' + a\beta\cos\xi \tag{12}$$

$$(a \cos\xi)\epsilon_\theta = u \tag{13}$$

$$a K_\xi = \beta' \tag{14}$$

$$(a \cos\xi)\, K_\theta = -\beta \sin\xi \tag{15}$$

These equations give rise to the two compatibility equations:

$$(\epsilon_\theta \cos\xi)' + \epsilon_\xi \sin\xi = \beta\cos\xi\ , \tag{16}$$

$$K_\xi + (K_\theta \cot\xi)' = 0\ . \tag{17}$$

The application of the uniformly distributed shearing force H_0 and bending moment M_0 to the equatorial edge ($\xi = 0$) of the hemispherical shell and the absence of stresses along the top edge ($\xi = \xi_1$) imply the following boundary conditions.

$$\xi = 0: \quad H = H_o \ , \ M_\xi = M_o \tag{18}$$

$$\xi = \xi_1: \quad H = M_\xi = 0 \tag{19}$$

Alternately, the conditions at the equatorial edge can be described by specifying the linear and angular displacements there.

$$\xi = 0: \quad u = u_o \ , \ \beta = \beta_o \tag{20}$$

The strain energy stored in the deformed hemispherical shell is designated by π_{STR} and assumes the form:

$$\pi_{STR} = (1/2)a^2 \int_0^{\xi_1}\int_0^{2\pi} (\epsilon_\xi N_\xi + \epsilon_\theta N_\theta + K_\xi M_\xi \tag{21}$$

$$+ K_\theta M_\theta)(\cos \xi) \, d\theta \, d\xi \ .$$

This expression may be transformed by substituting the equivalents of K_ξ, ϵ_θ, and N_θ as given by (6), (13), and (14) into (21) and performing an integration by parts. Use of the equilibrium equation (7) then gives:

$$\pi_{STR} = - \pi a \ (\beta_o M_o + u_o H_o) \ . \tag{22}$$

Alternate forms of Π_{STR} arising from simple substitutions are:

$$\Pi_{STR} = \pi a^2 \int_0^{\xi_1} [(N_\xi^2 - 2\nu N_\theta N_\xi + N_\theta^2)/C + (M_\xi^2 - 2\nu M_\theta M_\xi + M_\theta^2)/D(1-\nu^2)] \cos\xi \, d\xi \, , \tag{23}$$

or

$$\Pi_{STR} = \pi a^2 \int_0^{\xi_1} [(\epsilon_\xi^2 + 2\nu \epsilon_\xi \epsilon_\theta + \epsilon_\theta^2) C/(1-\nu^2) + D(K_\xi^2 + 2\nu K_\xi K_\theta + K_\theta^2)] \cos\xi \, d\xi \, . \tag{24}$$

CHAPTER III

DERIVATION OF THE BASIC INEQUALITIES

The principles of minimum complementary and minimum potential energy are now applied to suitable boundary-value problems. The results we seek are four basic inequalities. In the next chapter these four inequalities yield upper and lower bounds for the direct and inverse influence coefficients. The following problems are considered in order.

Case 1. Minimum complementary energy

$\xi = 0,\ H = H_0,\ M_\xi = M_0$

$\xi = \xi_1,\ H = M_\xi = 0$

Case 2. Minimum potential energy

$\xi = 0,\ H = H_0,\ M_\xi = M_0$

$\xi = \xi_1,\ H = M_\xi = 0$

Case 3. Minimum potential energy

$\xi = 0,\ u = u_0,\ \beta = \beta_0$

$\xi = \xi_1,\ H = M_\xi = 0$

Case 4. Minimum complementary energy

$\xi = 0,\ u = u_0,\ \beta = \beta_0$

$\xi = \xi_1,\ H = M_\xi = 0$

Cases 1 and 2 furnish upper and lower bounds for the direct influence coefficients; Cases 3 and 4 furnish upper and lower bounds for the inverse influence coefficients.

Case 1. Minimum Complementary Energy; Stress Boundary Conditions

Statement of the principle of minimum complementary energy.-- Let N_ξ, N_θ, M_ξ, M_θ be the actual stresses present in the deformed hemispherical shell. Then these stresses satisfy the basic relations and the stress boundary conditions specified below.

a. Equilibrium equations

$$N_\xi = - H \sin\xi \tag{25}$$

$$N_\theta = (H \cos\xi)' \tag{26}$$

$$(M_\xi \cos\xi)' + M_\theta \sin\xi + aH \cos^2\xi = 0 \tag{27}$$

b. Stress-displacement relations

$$(-u' + a\beta\cos\xi)/(a \sin\xi) \tag{28}$$

$$= (N_\xi - \nu N_\theta)/C$$

$$u/(a \cos\xi) = (N_\theta - \nu N_\xi)/C \tag{29}$$

$$\beta'/a = (M_\xi - \nu M_\theta)/D(1 - \nu^2) \tag{30}$$

$$- (\beta \sin\xi)/(a \cos\xi) \tag{31}$$

$$= (M_\theta - \nu M_\xi)/D(1 - \nu^2)$$

c. Compatibility equations

$$(\epsilon_\theta \cos\xi)' + \epsilon_\xi \sin\xi = \beta \cos\xi \ , \tag{32}$$

$$K_\xi + (K_\theta \cot\xi)' = 0 \ , \tag{33}$$

where the strains are defined by equations (12) through (15).

d. Boundary conditions

$$H(0) = H_o \ , \ M_\xi(0) = M_o \tag{34}$$

$$H(\xi_1) = M_\xi(\xi_1) = 0 \tag{35}$$

Let $\bar{N}_\xi$, $\bar{N}_\theta$, $\bar{M}_\xi$, $\bar{M}_\theta$ be the <u>varied stresses</u> satisfying the equilibrium equations and the (stress) boundary conditions.

a. Equilibrium equations

$$\bar{N}_\xi = -\bar{H} \sin\xi \tag{36}$$

$$\bar{N}_\theta = (\bar{H} \cos\xi)' \tag{37}$$

$$(\bar{M}_\xi \cos\xi)' + \bar{M}_\theta \sin\xi + a\bar{H} \cos^2\xi = 0 \tag{38}$$

b. Boundary conditions

$$\bar{H}(0) = H_o \ , \ \bar{M}_\xi(0) = M_o \tag{39}$$

$$\bar{H}(\xi_1) = \bar{M}_\xi(\xi_1) = 0 \tag{40}$$

We note that equation (36) offers a definition of the varied stress $\bar{H}$ in terms of $\bar{N}_\xi$ and assures the continuity of $\bar{H}$ (except perhaps at $\xi = 0$) if we assume continuity of the varied stress $\bar{N}_\xi$.

Let the <u>variations of the stresses</u> be defined as follows:

$$\bar{n}_\xi = \bar{N}_\xi - N_\xi \,, \tag{41}$$

$$\bar{n}_\theta = \bar{N}_\theta - N_\theta \,,$$

$$\bar{m}_\xi = \bar{M}_\xi - M_\xi \,,$$

$$\bar{m}_\theta = \bar{M}_\theta - M_\theta \,.$$

If the varied stresses are expressed in terms of the actual stresses and the stress variations and are then introduced into equations (36) through (40), the following relationships result.

a. Equilibrium equations

$$\bar{n}_\xi = - \bar{h} \sin\xi \tag{42}$$

$$\bar{n}_\theta = (\bar{h} \cos\xi)' \tag{43}$$

$$(\bar{m}_\xi \cos\xi)' + \bar{m}_\theta \sin\xi + a\bar{h} \cos^2\xi = 0 \tag{44}$$

b. Boundary conditions

$$\bar{h}(0) = \bar{m}_\xi(0) = \bar{h}(\xi_1) = \bar{m}_\xi(\xi_1) = 0 \qquad (45)$$

Again the equilibrium equation (42) defines the variation $\bar{h}$. If continuity is assumed for the varied stresses, then equation (41) assures the continuity of the variations of the stresses, and (42) assures the continuity of $\bar{h}$ (again, except perhaps at $\xi = 0$).

For the varied state of stress, the <u>complementary energy</u> of the hemispherical shell assumes the form:

$$\Pi_{sa}\,(\bar{N}_\xi\,,\ \bar{N}_\theta\,,\ \bar{M}_\xi\,,\ \bar{M}_\theta\,) \qquad (46)$$

$$= \pi\, a^2 \int_0^{\xi_1} [\,(\bar{N}_\xi^{\,2} + \bar{N}_\theta^{\,2} - 2\nu\,\bar{N}_\xi \bar{N}_\theta)/C \; + (\bar{M}_\xi^{\,2}$$

$$+ \bar{M}_\theta^{\,2} - 2\nu\,\bar{M}_\xi \bar{M}_\theta)/D(1 - \nu^2)]\ \cos\xi\ d\xi\ .$$

The principle of minimum complementary energy states that among all the varied states of stress that satisfy the equilibrium equations and the (stress) boundary conditions, the actual state of stress is such that it minimizes the complementary energy (46). This principle is summarized by the inequality below.

$$\Pi_{sa}\,(\bar{N}_\xi\,,\ \bar{N}_\theta\,,\ \bar{M}_\xi\,,\ \bar{M}_\theta\,)$$

$$\geq \Pi_{sa}\,(N_\xi\,,\ N_\theta\,,\ M_\xi\,,\ M_\theta\,)$$

<u>Proof of the principle of minimum complementary energy for the hemispherical shell</u>.--The definitions (41) are introduced into equation (46). A rearrangement of the terms then gives:

$$\pi_{sa}(\bar{N}_\xi, \bar{N}_\theta, \bar{M}_\xi, \bar{M}_\theta) \tag{47}$$

$$\left.\begin{aligned} = \quad & \pi a^2 \int_0^{\xi_1} [(N_\xi^2 + N_\theta^2 \\ & - 2\nu N_\xi N_\theta)/C + (M_\xi^2 + M_\theta^2 \\ & - 2\nu M_\xi M_\theta)/D(1-\nu^2)] \cos\xi \, d\xi \end{aligned}\right\} \quad (I)$$

$$\left.\begin{aligned} & + 2\pi a^2 \int_0^{\xi_1} [(N_\xi \bar{n}_\xi + N_\theta \bar{n}_\theta \\ & - \nu N_\xi \bar{n}_\theta - \nu N_\theta \bar{n}_\xi)/C \\ & + (M_\xi \bar{m}_\xi + M_\theta \bar{m}_\theta - \nu M_\xi \bar{m}_\theta \\ & - \nu M_\theta \bar{m}_\xi)/D(1-\nu^2)] \cos\xi \, d\xi \end{aligned}\right\} \quad (II)$$

$$\left.\begin{aligned} & + \pi a^2 \int_0^{\xi_1} [(\bar{n}_\xi^2 + \bar{n}_\theta^2 \\ & - 2\nu \bar{n}_\xi \bar{n}_\theta)/C + (\bar{m}_\xi^2 + \bar{m}_\theta^2 \\ & - 2\nu \bar{m}_\xi \bar{m}_\theta)/D(1-\nu^2)] \cos\xi \, d\xi \end{aligned}\right\} . (III)$$

As proof of the principle of minimum complementary energy it will now be shown that in equation (47):

$$(I) = \pi_{sa}(N_\xi, N_\theta, M_\xi, M_\theta) \; ;$$

$$(II) \equiv 0 \; ;$$

$$(III) \geq 0 \; .$$

It will then follow that:

$$\pi_{sa} (\bar{N}_{\xi} , \bar{N}_{\theta} , \bar{M}_{\xi} , \bar{M}_{\theta}) \geq \pi_{sa} (N_{\xi} , N_{\theta} , M_{\xi} , M_{\theta}) \; . \tag{48}$$

Comparison of term (I) of equation (47) with the definition of π_{sa}, equation (46), discloses that

$$(I) = \pi_{sa} (N_{\xi} , N_{\theta} , M_{\xi} , M_{\theta}) \; . \tag{49}$$

Consider term (II) of equation (47). Here we introduce the actual strains $\epsilon_{\xi}, \epsilon_{\theta}$, K_{ξ}, K_{θ} and eliminate the actual stresses N_{ξ}, N_{θ}, M_{ξ}, M_{θ} by use of equations (8) through (11). Thus

$$(II) = 2 \pi \; a^2 \int_0^{\xi_1} (\epsilon_{\theta} \bar{n}_{\theta} + \epsilon_{\xi} \bar{n}_{\xi} + K_{\xi} \bar{m}_{\xi} + K_{\theta} \bar{m}_{\theta}) \cos \xi \, d\xi \, . \tag{50}$$

The displacements u and β are introduced into (50) by use of equations (13) and (14), and the variation $\bar{n}_{\theta}$ is eliminated by use of equation (43). Then

$$(II) = 2\pi a^2 \int_0^{\xi_1} [\epsilon_\xi \bar{n}_\xi + K_\theta \bar{m}_\theta + \beta' \bar{m}_\xi / a \quad (51)$$

$$+ u (\bar{h} \cos\xi)'/(a \cos\xi)] \cos\xi \, d\xi .$$

It is now necessary to require continuity of the varied stresses. This insures that the variations of the stresses are continuous and allows the last two terms in the right member of equation (51) to be integrated by parts. Application of the boundary conditions (45) for the variations then gives:

$$(II) = 2\pi a^2 \int_0^{\xi_1} [\epsilon_\xi \bar{n}_\xi \cos\xi + K_\theta \bar{m}_\theta \cos\xi \quad (52)$$

$$- \bar{h}u' \cos\xi/a - \beta(\bar{m}_\xi \cos\xi)'/a] \, d\xi .$$

The relations (12), (14), and (42) are introduced into (52):

$$(II) = 2\pi a^2 \int_0^{\xi_1} - (\beta/a)[\bar{m}_\theta \sin\xi \quad (53)$$

$$(\bar{m}_\xi \cos\xi)' + a\bar{h} \cos^2\xi] \, d\xi .$$

The integrand of the integral on the right in equation (53) is seen to be zero by application of the equilibrium equation (44); hence

$$(II) \equiv 0 . \quad (54)$$

Consider term (III) of equation (47). We first note that

$$(\bar{n}_\xi - \bar{n}_\theta)^2 \geq 0 \ ;$$

hence

$$\bar{n}_\xi^{\,2} + \bar{n}_\theta^{\,2} \geq 2\,\bar{n}_\xi\,\bar{n}_\theta \ .$$

Since physical considerations demand that

$$0 \leq \nu \leq 1.0 \ ,$$

it follows that

$$\bar{n}_\xi^{\,2} + \bar{n}_\theta^{\,2} - 2\nu\,\bar{n}_\xi\,\bar{n}_\theta \geq 0 \tag{55}$$

and similarly that

$$\bar{m}_\xi^{\,2} + \bar{m}_\theta^{\,2} - 2\nu\,\bar{m}_\xi\,\bar{m}_\theta \geq 0 \ . \tag{56}$$

Inequalities (55) and (56) and the fact that

$$\cos\xi \geq 0 \qquad \text{for} \qquad \pi/2 \geq \xi_1 \geq \xi \geq 0$$

assure us that term (III) is greater than or equal to zero:

$$(III) \geq 0 \ . \tag{57}$$

Considering equation (47) in its entirety, statements (49), (54), and (57) allow the conclusion:

$$\Pi_{sa} (\overline{N}_\xi , \overline{N}_\theta , \overline{M}_\xi , \overline{M}_\theta) \tag{58}$$

$$\geq \Pi_{sa} (N_\xi , N_\theta , M_\xi , M_\theta) \ .$$

The principle of minimum complementary energy is indeed confirmed by the inequality (58); the complementary energy of the stressed hemispherical shell with a hole at the apex is a minimum when the varied stresses are equal to the actual stresses.

Derivation of the basic inequality 1.--Comparison of equations (46) and (23) indicates that

$$\Pi_{sa} (N_\xi , N_\theta , M_\xi , M_\theta) = \Pi_{STR} \ ; \tag{59}$$

hence, with reference to equations (22) and (58),

$$\Pi_{sa} (N_\xi , N_\theta , M_\xi , M_\theta) \tag{60}$$

$$\geq -\pi a \ (\beta_o M_o + u_o H_o) \ .$$

The definitions (1) of the direct influence coefficients (and the fact that $c_{uM} = c_{\beta H}$) are used to eliminate the displacements $\beta(0)$ and $u(0)$ from equation (60). The basic inequality 1 results:

$$\Pi_{sa}\,(\bar{N}_\xi\,,\ \bar{N}_\theta\,,\ \bar{M}_\xi\,,\ \bar{M}_\theta\,) \qquad (61)$$

$$\geq \pi a\ (c_{\beta M}\, M_o^{\,2} - 2\, c_{uM}\, M_o\, H_o + c_{uH}\, H_o^{\,2})\ .$$

Case 2. Minimum Potential Energy; Stress Boundary Conditions

Statement of the principle of minimum potential energy.--Let u and β be the actual displacements of the deformed hemispherical shell. These displacements then satisfy the basic relationships [equations (25) through (33)] and the (stress) boundary conditions.

Boundary conditions

$$H(0) = H_o\ ,\ M_\xi(0) = M_o \qquad (62)$$

$$H(\xi_1) = M_\xi(\xi_1) = 0$$

Let $\bar{u}$ and $\bar{\beta}$ be the varied displacements satisfying the compatibility equations but not necessarily the equilibrium equations nor the (stress) boundary conditions. Thus, $\bar{u}$ and $\bar{\beta}$ satisfy the following compatibility equations.

$$(\bar{\epsilon}_\theta \cos\xi)' + \bar{\epsilon}_\xi \sin\xi = \bar{\beta}\cos\xi \qquad (63)$$

$$\bar{K}_{\xi} + (\bar{K}_{\theta} \cot\xi)' = 0 \tag{64}$$

The varied strains appearing in the compatibility equations (63) and (64) are defined by the strain-displacement relations below.

Strain-displacement relations

$$\bar{\epsilon}_{\xi} = (-\bar{u}' + a\bar{\beta}\cos\xi)/(a \sin\xi) \tag{65}$$

$$\bar{\epsilon}_{\theta} = \bar{u}/(a \cos\xi) \tag{66}$$

$$\bar{K}_{\xi} = \bar{\beta}'/a \tag{67}$$

$$\bar{K}_{\theta} = (-\bar{\beta}\sin\xi)/(a \cos\xi) \tag{68}$$

Let the variations of the displacements be defined as follows.

$$\tilde{u} = \bar{u} - u \tag{69}$$

$$\tilde{\beta} = \bar{\beta} - \beta$$

These definitions (69) with equations (63) through (68) assure us that the variations $\tilde{u}$ and $\tilde{\beta}$ satisfy the compatibility equations when the variations of the strains are defined by the strain-displacement relations. Hence

$$(\tilde{\epsilon}_{\theta}\cos\xi)' + \tilde{\epsilon}_{\xi}\sin\xi = \tilde{\beta}\cos\xi \ , \tag{70}$$

$$\tilde{K}_\xi + (\tilde{K}_\theta \cot \xi)' = 0 \ , \tag{71}$$

where

$$\tilde{\epsilon}_\xi = (-\tilde{u}' + a\tilde{\beta}\cos\xi)/(a \sin\xi), \tag{72}$$

$$\tilde{\epsilon}_\theta = \tilde{u}/(a \cos\xi) \ , \tag{73}$$

$$\tilde{K}_\xi = \tilde{\beta}'/a \ , \tag{74}$$

$$\tilde{K}_\theta = (-\tilde{\beta}\sin\xi)/(a \cos\xi) \ . \tag{75}$$

We also note that if we assume continuity of the varied displacements $\bar{u}$ and $\bar{\beta}$, we are assured that the variations $\tilde{u}$ and $\tilde{\beta}$ are likewise continuous.

For the varied state of displacement, the potential energy of the distorted hemispherical shell assumes the form:

$$\begin{aligned} \Pi_{da}\ (\bar{u},\bar{\beta}\,) = \Pi a^2 \int_0^{\xi_1} [\,(\bar{\epsilon}_\xi{}^2 + \bar{\epsilon}_\theta{}^2 \\ + 2\nu\bar{\epsilon}_\xi\bar{\epsilon}_\theta)C/(1-\nu^2) + D\ (\bar{K}_\xi{}^2 \\ + \bar{K}_\theta{}^2 + 2\nu\bar{K}_\xi\bar{K}_\theta)]\cos\xi\ d\xi \\ + 2\Pi a\ [\bar{u}(0)\ H_0 + \bar{\beta}(0)\ M_0\ ]\ . \end{aligned} \tag{76}$$

The principle of minimum potential energy states that among all the varied states of displacements that satisfy the compatibility equations, the actual state of displacement is such that it minimizes the potential energy (76) of the distorted hemispherical shell.

The principle is summarized by the inequality below.

$$\Pi_{da}\ (\bar{u}, \bar{\beta}) \geq \Pi_{da}\ (u, \beta) \tag{77}$$

<u>Proof of the principle of minimum potential energy for the hemispherical shell</u>.--To facilitate computations, we introduce the definitions (69) into equations (65) through (68). We then apply equations (72) through (75) and obtain the following consequences of previous definitions.

$$\begin{aligned} \tilde{\epsilon}_{\xi} &= \bar{\epsilon}_{\xi} - \epsilon_{\xi} \\ \tilde{\epsilon}_{\theta} &= \bar{\epsilon}_{\theta} - \epsilon_{\theta} \\ \tilde{K}_{\xi} &= \bar{K}_{\xi} - K_{\xi} \\ \tilde{K}_{\theta} &= \bar{K}_{\theta} - K_{\theta} \end{aligned} \tag{78}$$

These equations are used to eliminate the varied strains from the integral term of equation (76). The constant term of (76) is transformed by use of (69). The resulting expression is:

$$
\begin{aligned}
&\pi_{da}\,(\bar{u},\bar{\beta}) && (79)\\
&= \pi a^2\int_0^{\xi_1} [(\epsilon_\xi{}^2 + \epsilon_\theta{}^2 \\
&\quad + 2\nu\epsilon_\xi\epsilon_\theta)C/(1-\nu^2) + (K_\xi{}^2 \\
&\quad + K_\theta{}^2 + 2\nu K_\xi K_\theta)D]\cos\xi\, d\xi && \text{(I)}\\
&\quad + 2\pi a^2\int_0^{\xi_1} [(\epsilon_\xi\tilde{\epsilon}_\xi + \epsilon_\theta\tilde{\epsilon}_\theta \\
&\quad + \nu\tilde{\epsilon}_\xi\epsilon_\theta + \nu\epsilon_\xi\tilde{\epsilon}_\theta)C/(1-\nu^2) \\
&\quad + (K_\xi\tilde{K}_\xi + K_\theta\tilde{K}_\theta + \nu K_\xi\tilde{K}_\theta \\
&\quad + \nu\tilde{K}_\xi K_\theta)D]\cos\xi\, d\xi && \text{(II)}\\
&\quad + \pi a^2\int_0^{\xi_1} [(\tilde{\epsilon}_\xi{}^2 + \tilde{\epsilon}_\theta{}^2 \\
&\quad + 2\nu\tilde{\epsilon}_\xi\tilde{\epsilon}_\theta)C/(1-\nu^2) + (\tilde{K}_\xi{}^2 \\
&\quad + \tilde{K}_\theta{}^2 + 2\nu\tilde{K}_\xi\tilde{K}_\theta)D]\cos\xi\, d\xi && \text{(III)}\\
&\quad + 2\pi a\,[u(0)\,H_o + \beta(0)\,M_o\,] && \text{(IV)}\\
&\quad + 2\pi a\,[\tilde{u}(0)\,H_o + \tilde{\beta}(0)\,M_o\,]\,. && \text{(V)}
\end{aligned}
$$

As proof of the principle of minimum potential energy, we will now show that in equation (79):

$$(I) + (IV) = \pi_{da}\,(u,\beta)\ ,$$

$$(II) = -\,(V)$$

$$(\mathrm{III}) \geq 0 \ .$$

It will then follow that:

$$\pi_{da}(\bar{u}, \bar{\beta}) \geq \pi_{da}(u, \beta) \ . \tag{80}$$

Comparison of the terms (I) and (IV) with the definition (76) of π_{da} discloses that

$$(\mathrm{I}) + (\mathrm{IV}) = \pi_{da}(u, \beta) \ . \tag{81}$$

Term (II) of (79) is transformed by the stress-strain relations (8) through (11) into:

$$(\mathrm{II}) = 2\pi a^2 \int_0^{\xi_1} (\tilde{\epsilon}_\xi N_\xi + \tilde{\epsilon}_\theta N_\theta + \tilde{K}_\theta M_\theta + \tilde{K}_\xi M_\xi) \cos\xi \, d\xi \ . \tag{82}$$

The variations of the strains $\tilde{\epsilon}_\theta$ and $\tilde{K}_\xi$ are defined by equations (73) and (74). These two quantities are eliminated from (82), as is the actual stress N_θ, by use of equations (26), (73), and (74). Thus, (82) becomes:

$$(\mathrm{II}) = 2\pi a^2 \int_0^{\xi_1} [\tilde{\epsilon}_\xi N_\xi + \tilde{K}_\theta M_\theta + \tilde{u}\,(H \cos\xi)'/(a\cos\xi) + (1/a)\,\tilde{\beta}'\, M_\xi] \cos\xi \, d\xi \ . \tag{83}$$

The proof is now restricted to only those cases in which the varied displacements $\bar{u}$ and $\bar{\beta}$ are continuous. The last two terms of the right side of (83) are integrated by parts. Use of the stress boundary conditions then gives:

$$(II) = -2 \pi a [\tilde{u}(0) H_o + \tilde{\beta}(0) M_o] . \quad (84)$$

Hence,

$$(II) = - (V) , \quad (85)$$

and

$$(II) + (V) = 0 . \quad (86)$$

Examination of term (III) of equation (79) discloses:

$$(III) \geq 0 . \quad (87)$$

Considering equation (79) in its entirety, statements (81), (85), and (87) allow the conclusion:

$$\pi_{da} (\bar{u}, \bar{\beta}) \geq \pi_{da} (u, \beta) . \quad (88)$$

The principle of minimum potential energy is confirmed by the inequality (88); the potential energy of the distorted hemispherical shell with a hole in the apex is a minimum when the varied displacements are equal to the actual displacements.

<u>Derivation of the basic inequality 2</u>.--Comparison of equations (76) and (24) indicates that

$$\pi_{da}\ (u, \beta) \tag{89}$$

$$= \pi_{STR} - 2\,\pi\, a\,[u(0)\, H_o + \beta(0)\, M_o]\ .$$

This result, when entered into (88), gives:

$$\pi_{da}\ (\bar{u}, \bar{\beta}) \geq \pi a\,[u(0)\, H_o + \beta(0)\, M_o]\ . \tag{90}$$

The definitions (1) of the direct influence coefficients are used to eliminate the displacements u(0) and $\beta(0)$ from (90).

$$\pi_{da}\ (\bar{u}, \bar{\beta}) \geq \pi a\ (-c_{\beta M}\, M_o^{\ 2} \tag{91}$$

$$+ 2\, c_{uM}\, M_o\, H_o - c_{uH}\, H_o^{\ 2})\ .$$

When signs are changed and the sense of the inequality (91) is reversed, the <u>basic inequality 2</u> results:

$$-\pi_{da}\ (\bar{u}, \bar{\beta}) \leq \pi a\ (c_{\beta M}\, M_o^{\ 2} \tag{92}$$

$$- 2\, c_{uM}\, M_o\, H_o + c_{uH}\, H_o^{\ 2})\ .$$

Case 3. Minimum Potential Energy; Mixed Boundary Conditions

Statement of the principle of minimum potential energy.--Let u and β be the actual displacements of the deformed hemispherical shell. These displacements then satisfy the basic relationships [equations (25) through (33)] and the boundary conditions below.

$$u(0) = u_0 \ , \ \beta(0) = \beta_0 \tag{93}$$

$$H(\xi_1) = M_\xi(\xi_1) = 0 \tag{94}$$

Let $\bar{u}$ and $\bar{\beta}$ be the varied displacements satisfying the compatibility equations and the displacement boundary conditions but not necessarily the equilibrium equations nor the stress boundary conditions. Thus, $\bar{u}$ and $\bar{\beta}$ satisfy the following relations.

Compatibility equations

$$(\bar{\epsilon}_\theta \cos\xi)' + \bar{\epsilon}_\xi \sin\xi = \bar{\beta}\cos\xi \tag{95}$$

$$\bar{K}_\xi + (\bar{K}_\theta \cot\xi)' = 0 \tag{96}$$

Displacement boundary conditions

$$\bar{u}(0) = u_0 \ , \bar{\beta}(0) = \beta_0 \tag{97}$$

The varied strains appearing in equations (95) and (96) are defined as in Case 1, equations (65) through (68).

Let the <u>variations</u> of the displacements be defined as follows.

$$\tilde{u} = \bar{u} - u \tag{98}$$

$$\tilde{\beta} = \bar{\beta} - \beta$$

These definitions, taken with equations (95), (96), and (97), assure us that the variations $\tilde{u}$ and $\tilde{\beta}$ themselves satisfy the compatibility equations and the displacement boundary conditions.

Compatibility equations

$$(\tilde{\epsilon}_\theta \cos\xi) + \tilde{\epsilon}_\xi \sin\xi = \tilde{\beta}\cos\xi \tag{99}$$

$$\tilde{K}_\xi + (\tilde{K}_\theta \cot\xi)' = 0 \tag{100}$$

Boundary conditions

$$\tilde{u}(0) = \tilde{\beta}(0) = 0 \tag{101}$$

The variations of the strains appearing in equations (99) and (100) are as defined in Case 1, equations (72) through (75).

For the varied state of displacement, the potential energy of the distorted hemispherical shell assumes the form:

$$\Pi_{db}\ (\bar{u}, \bar{\beta}) = \pi a^2 \int_0^{\xi_1} (\bar{\epsilon}_\xi{}^2 + \bar{\epsilon}_\theta{}^2 \tag{102}$$

$$+ 2\nu \bar{\epsilon}_\xi \bar{\epsilon}_\theta) C/(1 - \nu^2) + D(\bar{K}_\xi{}^2$$

$$+ \bar{K}_\theta{}^2 + 2\nu \bar{K}_\xi \bar{K}_\theta)]\ \cos\xi\ d\xi .$$

The principle of minimum potential energy states that among all the varied states of displacements satisfying equations (95), (96), and (97), the actual state of displacement minimizes the potential energy (102) of the distorted hemispherical shell. This principle is summarized by the inequality

$$\Pi_{db}\ (\bar{u}, \bar{\beta}) \geq \Pi_{db}\ (u, \beta)\ . \tag{103}$$

The proof in this case closely parallels the proof presented in Case 2. Hence, accepting the validity of (103), the basic inequality 3 is immediately derived.

Derivation of the basic inequality 3.---Comparison of the definition (102) with equations (24) and (22) indicates that

$$\Pi_{db}(u, \beta) = \Pi_{STR} = -\ \pi\ a\ (\beta_o\ M_o + u_o\ H_o)\ . \tag{104}$$

Hence, inequality (103) yields:

$$\pi_{db}(\bar{u},\bar{\beta}) \geq -\pi a\,[\beta(0)\,M_o + u(0)\,H_o]\ . \tag{105}$$

The definitions (2) of the inverse influence coefficients are used to eliminate the stresses M_o and H_o from the inequality (105). The basic inequality 3 is the result:

$$\pi_{db}(\bar{u},\bar{\beta}) \geq \pi a\,(k_{uH}\,u_o{}^2 + 2\,k_{uM}\,u_o\,\beta_o + k_{\beta M}\,\beta_o{}^2)\ . \tag{106}$$

Case 4. Minimum Complementary Energy; Mixed Boundary Conditions

Statement of the principle of minimum complementary energy.--Let N_ξ, N_θ, M_ξ, M_θ be the actual stresses present in the stressed hemispherical shell. Then these stresses satisfy the basic relationships [equations (25) through (33)] and the following boundary conditions.

Boundary conditions

$$u(0) = u_o\ ,\ \beta(0) = \beta_o \tag{107}$$

$$H(\xi_1) = M_\xi(\xi_1) = 0 \tag{108}$$

Let $\bar{N}_\xi$, $\bar{N}_\theta$, $\bar{M}_\xi$, $\bar{M}_\theta$ be the varied stresses satisfying the equilibrium equations and the stress boundary conditions at $\xi = \xi_1$. Thus, the varied stresses satisfy the following.

Equilibrium equations

$$\bar{N}_{\xi} = -\bar{H}\sin\xi \tag{109}$$

$$\bar{N}_{\theta} = (\bar{H}\cos\xi)' \tag{110}$$

$$(\bar{M}_{\xi}\cos\xi)' + \bar{M}_{\theta}\sin\xi + a\bar{H}\cos^2\xi = 0 \tag{111}$$

Boundary conditions

$$\bar{H}(\xi_1) = \bar{M}_{\xi}(\xi_1) = 0 \tag{112}$$

Let the <u>variations of the stresses</u> be defined as below.

$$\begin{aligned} \bar{n}_{\xi} &= \bar{N}_{\xi} - N_{\xi} \\ \bar{n}_{\theta} &= \bar{N}_{\theta} - N_{\theta} \\ \bar{m}_{\xi} &= \bar{M}_{\xi} - M_{\xi} \\ \bar{m}_{\theta} &= \bar{M}_{\theta} - M_{\theta} \end{aligned} \tag{113}$$

These variations of the stresses are found to satisfy the same requirements as the varied stresses.

Equilibrium equations

$$\bar{n}_\xi = -\bar{h}\sin\xi \tag{114}$$

$$\bar{n}_\theta = (\bar{h}\cos\xi)' \tag{115}$$

$$(\bar{m}_\xi\cos\xi)' + \bar{m}_\theta\sin\xi + a\bar{h}\cos^2\xi = 0 \tag{116}$$

Boundary conditions

$$\bar{h}(\xi_1) = \bar{m}_\xi(\xi_1) = 0 \tag{117}$$

For the varied state of stress, the complementary energy of the hemispherical shell assumes the form:

$$\begin{aligned} \Pi_{sb}(\bar{N}_\xi, \bar{N}_\theta, \bar{M}_\xi, \bar{M}_\theta) &= \pi a^2 \int_0^{\xi_1} [(\bar{N}_\xi^2 \\ &+ \bar{N}_\theta^2 - 2\nu\bar{N}_\xi\bar{N}_\theta)/C + (\bar{M}_\xi^2 + \bar{M}_\theta^2 \\ &- 2\nu\bar{M}_\xi\bar{M}_\theta)/D(1-\nu^2)]\cos\xi\, d\xi \\ &+ 2\pi a[u_0\bar{H}(0) + \beta_0\bar{M}_\xi(0)] . \end{aligned} \tag{118}$$

The principle of minimum complementary energy states that among all the varied states of stress that satisfy equations (109) through (112), the actual state of stress is such that it minimizes the complementary energy (118). This principle is summarized by the inequality below.

$$\begin{aligned} &\Pi_{sb}(\bar{N}_\xi, \bar{N}_\theta, \bar{M}_\xi, \bar{M}_\theta) \\ &\geq \Pi_{sb}(N_\xi, N_\theta, M_\xi, M_\theta) \end{aligned} \tag{119}$$

The proof in this case closely parallels the proof given in Case 1. Hence, accepting the validity of (119), the basic inequality 4 is immediately derived.

Derivation of the basic inequality 4.--Comparison of the definition (118) with (23) and (22) indicates that

$$\pi_{sb}\,(N_\xi\,,\ N_\theta\,,\ M_\xi\,,\ M_\theta\,) \qquad (120)$$

$$= \pi a\ (\beta_o\ M_o + u_o\ H_o)\ .$$

Hence, the inequality (119) yields

$$\pi_{sb}\,(\bar{N}_\xi\,,\ \bar{N}_\theta\,,\ \bar{M}_\xi\,,\ \bar{M}_\theta\,) \qquad (121)$$

$$\geq \pi a\ [u_o\ H_o + \beta_o\ M_o]\ .$$

The definitions (2) of the inverse influence coefficients are used to eliminate the stresses M_o and H_o from the inequality (121). The inequality is then reversed by changing signs, and the basic inequality 4 results:

$$-\ \pi_{sb}\,(\bar{N}_\xi\,,\ \bar{N}_\theta\,,\ \bar{M}_\xi\,,\ \bar{M}_\theta\,) \qquad (122)$$

$$\leq \pi a\ (k_{uH}\ u_o{}^2 + 2\ k_{uM}\ u_o\ \beta_o + k_{\beta M}\ \beta_o{}^2)\ .$$

CHAPTER IV

DERIVATION OF THE BOUNDS FOR THE INFLUENCE COEFFICIENTS

Upper and lower bounds for the direct and inverse influence coefficients for the thin hemispherical shell with a hole at the apex are now derived. The bounds are obtained from the four basic inequalities derived in Chapter III.

Transformation of the Basic Inequality 1

Choice of the varied stresses.--The basic inequality 1, (61), is repeated below.

$$\pi a^2 \int_0^{\xi_1} [\bar{N}_\xi{}^2 + \bar{N}_\theta{}^2 - 2\nu \bar{N}_\xi \bar{N}_\theta)/C + (\bar{M}_\xi{}^2 + \bar{M}_\theta{}^2 - 2\nu \bar{M}_\xi \bar{M}_\theta)/D(1-\nu^2)] \cos\xi \, d\xi \geq \pi a \, (c_{\beta M} M_o{}^2 - 2\, c_{uM} M_o H_o + c_{uH} H_o{}^2) \tag{123}$$

It is noted that the varied stresses appearing in (123) must satisfy the equilibrium equations and the (stress) boundary conditions of Case 1, Chapter III. Hence $\bar{N}_\xi$, $\bar{N}_\theta$, $\bar{M}_\xi$, and $\bar{M}_\theta$ must satisfy the following relations.

a. Equilibrium equations

$$\overline{N}_{\xi} = -\overline{H}\sin\xi \tag{124}$$

$$\overline{N}_{\theta} = (\overline{H}\cos\xi)' \tag{125}$$

$$(\overline{M}_{\xi}\cos\xi)' + \overline{M}_{\theta}\sin\xi + a\overline{H}\cos^2\xi = 0 \tag{126}$$

b. Boundary conditions

$$\overline{H}(0) = H_o\ ,\ \overline{M}_{\xi}(0) = M_o \tag{127}$$

$$\overline{H}(\xi_1) = \overline{M}_{\xi}(\xi_1) = 0 \tag{128}$$

Also, the proof given for the principle of minimum complementary energy required that the varied stresses be continuous.

Let the varied stresses $\overline{H}$ and $\overline{M}_{\xi}$ be chosen as below.

$$\overline{M}_{\xi} = [-a\ H_o\ g_{1a}(\xi) + M_o\ g_{2a}(\xi)]/\cos\xi \tag{129}$$

$$\overline{H} = [H_o\ g_{3a}(\xi) - (1/a)\ M_o\ g_{4a}(\xi)]/\cos\xi \tag{130}$$

The functions g_{1a}, g_{2a}, g_{3a}, g_{4a} are dimensionless and are arbitrary except for the following requirements.

a. Equilibrium equations: The varied stresses must satisfy the equilibrium equations (124), (125) and (126). These equations will be satisfied if they are used to determine $\overline{N}_{\xi}$, $\overline{N}_{\theta}$ and $\overline{M}_{\theta}$ in terms of $\overline{H}$ and $\overline{M}_{\xi}$. Thus

$$\bar{N}_{\xi} = -\bar{H}\sin\xi\ , \tag{131}$$

$$\bar{N}_{\theta} = (\bar{H}\cos\xi)'\ , \tag{132}$$

$$\bar{M}_{\theta} = [(\bar{M}_{\xi}\cos\xi)' + a\bar{H}\cos^2\xi]/\sin\xi\ . \tag{133}$$

To insure that $\bar{M}_{\theta}$, as defined by (133), is finite at $\xi = 0$ we require that

$$(\bar{M}_{\xi}\cos\xi)' + a\bar{H}\cos^2\xi = O(\sin\xi) \text{ as } \xi \to 0\ . \tag{134}$$

In terms of g_{1a}, g_{2a}, g_{3a}, and g_{4a} this condition takes the form:

$$\left.\begin{array}{l} g'_{1a}(\xi) - g_{3a}(\xi)\cos\xi \\ g'_{2a}(\xi) - g_{4a}(\xi)\cos\xi \end{array}\right\} = O(\sin\xi) \text{ as } \xi \to 0\ . \tag{135}$$

b. Boundary conditions: The boundary conditions, (127) and (128), must be satisfied. Hence,

$$M_o = -a\,H_o\,g_{1a}(0) + M_o\,g_{2a}(0)\ , \tag{136}$$

$$H_o = H_o\,g_{3a}(0) - (1/a)\,M_o\,g_{4a}(0)\ , \tag{137}$$

and

$$-a\,H_o\,g_{1a}(\xi_1) + M_o\,g_{2a}(\xi_1) = 0\ , \tag{138}$$

$$H_o \, g_{3a}(\xi_1) - (1/a)\, M_o \, g_{4a}(\xi_1) = 0 \, . \tag{139}$$

c. Continuity: The functions $g_{1a}(\xi)$, $g_{2a}(\xi)$, $g_{3a}(\xi)$, and $g_{4a}(\xi)$ must be such that $\overline{M}_\xi$ and $\overline{H}$, as defined by (129) and (130), are continuous.

The above restrictions a., b., and c. are sufficient to insure that the chosen varied stresses satisfy the requirements needed for the validity of the <u>basic inequality 1</u>.

<u>Substitution into the basic inequality 1</u>.--The varied stresses appearing in (123) are expressed in terms of H_o, M_o, $g_{1a}(\xi)$, $g_{2a}(\xi)$, $g_{3a}(\xi)$, $g_{4a}(\xi)$, and these values are substituted into (123). After a rearrangement of terms, the <u>basic inequality 1</u> takes the form:

$$\begin{aligned}(a^3/2\, D\lambda^3)\, G_{1a}\, H_o^{\,2} - 2(a^2/2\, D\lambda^2)\, G_{12a}\, H_o\, M_o \\ + (a/D\lambda)\, G_{2a}\, M_o^{\,2} \geq c_{uH}\, H_o^{\,2} \\ - 2\, c_{uM}\, H_o\, M_o + c_{\beta M}\, M_o^{\,2} \, .\end{aligned} \tag{140}$$

The functions G_{1a}, G_{12a}, and G_{2a} are as follows:

$$\begin{aligned}G_{1a} = \int_0^{\xi_1} \{ [g_{3a}^{\,2} \tan\xi \, \sin\xi + (g'_{3a})^2 \cos\xi \\ 2\nu\, g_{3a}\, g'_{3a} \sin\xi]/2\lambda + [g_{1a}^{\,2} \sec\xi \\ (g'_{1a})^2 \cot\xi \csc\xi - 2\, g'_{1a}\, g_{3a} \cot^2\xi\end{aligned} \tag{141}$$

$$+ g_{3a}^{\;2} \cot^2\xi \cos\xi + 2\nu g_{1a} g'_{1a} \csc\xi$$

$$- 2\nu g_{1a} g_{3a} \cot\xi \,]\, 2\lambda^3/(1-\nu^2) \Big\} \, d\xi ,$$

$$G_{12a} = \int_0^{\xi_1} \Big\{ [g_{3a} g_{4a} \tan\xi \sin\xi \tag{142}$$

$$+ (g'_{3a} g_{4a} + g_{3a} g'_{4a}) \nu \sin\xi$$

$$+ g'_{3a} g'_{4a} \cos\xi]/2\lambda^2$$

$$+ [g_{1a} g_{2a} \sec\xi + g'_{1a} g'_{2a} \cot\xi \csc\xi$$

$$- (g'_{2a} g_{3a} + g'_{1a} g_{4a}) \cot^2\xi$$

$$+ \nu (g'_{1a} g_{2a} + g_{1a} g'_{2a}) \csc\xi$$

$$- \nu (g_{1a} g_{4a} + g_{2a} g_{3a}) \cot\xi$$

$$+ g_{3a} g_{4a} \cot^2\xi \cos\xi \,]\, 2\lambda^2/(1-\nu^2) \Big\} \, d\xi ,$$

$$G_{2a} = \int_0^{\xi_1} \Big\{ [g_{4a}^{\;2} \tan\xi \sin\xi + (g'_{4a})^2 \cos\xi \tag{143}$$

$$+ 2\nu g_{4a} g'_{4a} \sin\xi \,]/4\lambda^3 + [g_{2a}^{\;2} \sec\xi$$

$$+ (g'_{2a})^2 \cot\xi \csc\xi - 2 g'_{2a} g_{4a} \cot^2\xi$$

$$+ g_{4a}^{\;2} \cot^2\xi \cos\xi + 2\nu g_{2a} g'_{2a} \csc\xi$$

$$- 2\nu g_{2a} g_{4a} \cot\xi \,]\, \lambda/(1-\nu^2) \Big\} \, d\xi .$$

Transformation of the Basic Inequality 2

<u>Choice of the varied displacements</u>.--The basic inequality 2, (92), is repeated below.

$$
\begin{aligned}
&-\pi a^2\int_0^{\xi_1}[(\bar{\epsilon}_\xi{}^2 + \bar{\epsilon}_\theta{}^2 + 2\nu\bar{\epsilon}_\xi\bar{\epsilon}_\theta)C/(1-\nu^2) \\
&\quad + D(\bar{K}_\xi{}^2 + \bar{K}_\theta{}^2 + 2\nu\bar{K}_\xi\bar{K}_\theta)]\cos\xi\, d\xi \\
&\quad - 2\pi a\,[\bar{u}(0)\,H_o + \bar{\beta}(0)\,M_o] \\
&\quad \leq \pi a(c_{\beta M}\,M_o{}^2 - 2\,c_{uM}\,M_o\,H_o + c_{uH}\,H_o{}^2)\ .
\end{aligned}
\tag{144}
$$

It is noted that the varied displacements, $\bar{u}$ and $\bar{\beta}$, must satisfy only the compatibility equations. Thus

$$(\bar{\epsilon}_\theta\cos\xi)' + \bar{\epsilon}_\xi\sin\xi = \bar{\beta}\cos\xi\ , \tag{145}$$

$$\bar{K}_\xi + (\bar{K}_\theta\cot\xi)' = 0\ . \tag{146}$$

Any chosen displacements will satisfy these equations if the varied strains appearing in (145) and (146) are defined by the strain-displacement relations which follow.

Strain-displacement relations

$$\bar{\epsilon}_\xi = (-\bar{u}' + a\bar{\beta}\cos\xi)/(a\sin\xi) \tag{147}$$

$$\bar{\epsilon}_\theta = \bar{u}/(a\cos\xi) \tag{148}$$

$$\bar{K}_{\xi} = \bar{\beta}'/a \tag{149}$$

$$\bar{K}_{\theta} = (-\bar{\beta}\sin\xi)/(a\cos\xi) \tag{150}$$

Also, the proof given for the principle of minimum potential energy required that the varied displacements be continuous.

Let the varied displacements $\bar{u}$ and $\bar{\beta}$ be chosen as below.

$$\bar{u}(\xi) = (a/C)\, H_o\, f_{1a}(\xi) + (a^2/D)\, M_o\, f_{2a}(\xi) \tag{151}$$

$$\bar{\beta}(\xi) = (1/C)\, H_o\, f_{3a}(\xi) + (a/D)\, M_o\, f_{4a}(\xi) \tag{152}$$

The functions $f_{1a}(\xi)$, $f_{2a}(\xi)$, $f_{3a}(\xi)$, and $f_{4a}(\xi)$ are dimensionless and are arbitrary except for the following requirements.

a. Compatibility equations: The varied displacements $\bar{u}$ and $\bar{\beta}$ must satisfy the compatibility equations. This requires that the varied strains be defined by equations (147) through (150). To insure that $\bar{\epsilon}_{\xi}$ is finite at $\xi = 0$, we must have

$$-\bar{u}' + a\bar{\beta}\cos\xi = 0\ (\sin\xi) \text{ as } \xi \rightarrow 0. \tag{153}$$

In terms of f_{1a}, f_{2a}, f_{3a}, and f_{4a}, this condition takes the form:

$$\left.\begin{aligned} f_{3a}(\xi)\cos\xi + f_{1a}'(\xi) \\ f_{4a}(\xi)\cos\xi + f_{2a}'(\xi) \end{aligned}\right\} = O\ (\sin\xi)\ \text{as}\ \xi \to 0\ . \tag{154}$$

b. Continuity: The functions f_{1a}, f_{2a}, f_{3a}, and f_{4a} must be such that the varied displacements, as defined by (151) and (152), are continuous.

The two restrictions above, a. and b., are sufficient to insure that the chosen varied displacements satisfy the requirements needed for the validity of the <u>basic inequality 2</u>.

<u>Substitution into the basic inequality 2</u>.--The varied strains appearing in (144) are expressed in terms of H_o, M_o, and f_{1a}, f_{2a}, f_{3a}, f_{4a} by introducing $\bar{u}$ and $\bar{\beta}$, (151) and (152), into the definitions (147) through (150). After a rearrangement of terms, (144) takes the form

$$\begin{aligned} &(a^3/2\,D\lambda^3)\,F_{1a}\,H_o^{\,2} - 2(a^2/2\,D\lambda^2)\,F_{12a}\,H_o\,M_o \\ &\quad + (a/D\lambda)\cdot F_{2a}\,M_o^{\,2} \le c_{uH}\,H_o^{\,2} \\ &\quad - 2\,c_{uM}\,H_o\,M_o + c_{\beta M}\,M_o^{\,2}\ . \end{aligned} \tag{155}$$

The functions F_{1a}, F_{12a}, and F_{2a} take the form shown below.

$$\begin{aligned} F_{1a} = &- (1/\lambda)\ f_{1a}(0) - \int_0^{\xi_1} \{ [f_{1a}^{\,2} \sec\xi \\ &- 2\,f_{1a}'\,f_{3a}\cot^2\xi + 2\nu\,f_{1a}\,f_{3a}\cot\xi \end{aligned} \tag{156}$$

$$+ f_{3a}{}^2 \cot^2\xi \cos\xi - 2\nu f_{1a} f'_{1a} \csc\xi$$

$$+ (f'_{1a})^2 \cot\xi \csc\xi]/2\lambda(1-\nu^2)$$

$$+ [(f'_{3a})^2 \cos\xi + f_{3a}{}^2 \tan\xi \sin\xi$$

$$- 2\nu f'_{3a} f_{3a} \sin\xi]\ 2\lambda^2 \Big\}\ d\xi$$

$$F_{12a} = 2\lambda^2 f_{2a}(0) + (1/2\lambda^2)\ f_{3a}(0) \tag{157}$$

$$+\int_0^{\xi_1} \Big\{[\nu(f_{1a} f_{4a} + f_{2a} f_{3a}) \cot\xi$$

$$+ f'_{1a} f'_{2a} \cot\xi \csc\xi + f_{1a} f_{2a} \sec\xi$$

$$- \nu(f'_{1a} f_{2a} + f_{1a} f'_{2a}) \csc\xi$$

$$- (f'_{1a} f_{4a} + f'_{2a} f_{3a}) \cot^2\xi$$

$$+ f_{3a} f_{4a} \cot^2\xi \cos\xi]2\lambda^2/(1-\nu^2)$$

$$+ [f'_{3a} f'_{4a} \cos\xi + f_{3a} f_{4a} \tan\xi \sin\xi$$

$$+ \nu(f'_{3a} f_{4a} + f_{3a} f'_{4a}) \sin\xi]/2\lambda^2 \Big\}\ d\xi$$

$$F_{2a} = -2\lambda f_{4a}(0) - \int_0^{\xi_1} \Big\{[2\nu f_{2a} f_{4a} \cot\xi \tag{158}$$

$$+ (f'_{2a})^2 \cot\xi \csc\xi - 2 f'_{2a} f_{4a} \cot^2\xi$$

$$+ f_{4a}{}^2 \cot^2\xi \cos\xi - 2\nu f_{2a} f'_{2a} \csc\xi$$

$$+ f_{2a}{}^2 \sec\xi]4\lambda^5/(1-\nu^2) + [(f'_{4a})^2 \cos\xi$$

$$+ f_{4a}{}^2 \tan\xi \sin\xi - 2\nu f_{4a} f'_{4a} \sin\xi]\lambda \Big\} d\xi$$

Transformation of the Basic Inequality 3

Choice of the varied displacements.--The basic inequality 3, (106), is repeated below.

$$\pi a^2 \int_0^{\xi_1} [(\bar{\epsilon}_{\xi}^2 + \bar{\epsilon}_{\theta}^2 + 2\nu \bar{\epsilon}_{\xi} \bar{\epsilon}_{\theta}) C/(1 - \nu^2) \tag{159}$$

$$+ D \ (\bar{K}_{\xi}^2 + \bar{K}_{\theta}^2 + 2\nu \bar{K}_{\xi} \bar{K}_{\theta})] \cos\xi \ d\xi$$

$$\geq \pi a \ (k_{uH} \ u_o^2 + 2 \ k_{uM} \ u_o \ \beta_o + k_{\beta M} \ \beta_o^2) \ .$$

It is noted that the varied displacements, $\bar{u}$ and $\bar{\beta}$, must satisfy the compatibility equations and the displacement boundary conditions specified at $\xi = 0$. Thus $\bar{u}$ and $\bar{\beta}$ satisfy the following.

Compatibility equations

$$(\bar{\epsilon}_{\theta} \cos\xi)' + \bar{\epsilon}_{\xi} \sin\xi = \bar{\beta} \cos\xi \tag{160}$$

$$\bar{K}_{\xi} + (\bar{K}_{\theta} \cot\xi)' = 0$$

Boundary conditions

$$\bar{u}(0) = u_o \ , \ \bar{\beta}(0) = \beta_o \tag{161}$$

Also, the proof given for the principle of minimum potential energy required that the varied displacements be continuous.

Let the varied displacements $\bar{u}$ and $\bar{\beta}$ be chosen as below.

$$\bar{u}(\xi) = u_o\ f_{1b}(\xi) + a\,\beta_o\ f_{2b}(\xi) \tag{162}$$

$$\bar{\beta}(\xi) = (1/a)\ u_o\ f_{3b}(\xi) + \beta_o\ f_{4b}(\xi)$$

The functions f_{1b}, f_{2b}, f_{3b}, and f_{4b} are dimensionless and are arbitrary except for the following requirements. Since these requirements are similar to those presented in "Transformation of the Basic Inequality 2," their derivation is not shown.

a. Compatibility equations

$$\left.\begin{array}{l} f_{3b}(\xi)\ \cos\xi - f'_{1b}(\xi) \\ f_{4b}(\xi)\ \cos\xi - f'_{2b}(\xi) \end{array}\right\} = 0\ (\sin\xi)\ \text{as}\ \xi \rightarrow 0 \tag{163}$$

b. Boundary conditions

$$u_o\ f_{1b}(0) + a\,\beta_o\ f_{2b}(0) = u_o \tag{164}$$

$$(1/a)\ u_o\ f_{3b}(0) + \beta_o\ f_{4b}(0) = \beta_o$$

c. Continuity: The functions f_{1b}, f_{2b}, f_{3b}, and f_{4b} must be so chosen that the displacements $\bar{u}$ and $\bar{\beta}$, (162), are continuous.

Substitution into the basic inequality 3.--The varied strains appearing in (159) are expressed in terms of u_o, β_o, and f_{1b}, f_{2b}, f_{3b}, f_{4b} by introducing $\bar{u}$ and $\bar{\beta}$, (162), into the definitions (147) through (150). After a rearrangement of terms, (159) takes the form

$$\begin{aligned}(4D\lambda^3/a^3)\, F_{1b}\, u_o^{\,2} + 2\,(2D\lambda^2/a^2)\, F_{12b}\, u_o\, \beta_o \\ + (2D\lambda/a)\, F_{2b}\, \beta_o^{\,2} \geq k_{uH}\, u_o^{\,2} \\ + 2\, k_{uM}\, u_o\, \beta_o + k_{\beta M}\, \beta_o^{\,2}\,,\end{aligned} \tag{165}$$

where the quantities F_{1b}, F_{12b}, and F_{2b} are given by expressions similar to equations (156), (157), and (158).

Transformation of the Basic Inequality 4

Choice of the varied stresses.--The basic inequality 4, (122), is repeated below.

$$\begin{aligned}-\pi a^2 \int_0^{\xi_1} [\,(\bar{N}_\xi^{\,2} + \bar{N}_\theta^{\,2} - 2\nu\, \bar{N}_\xi\, \bar{N}_\theta)/C + (\bar{M}_\xi^{\,2} \\ + \bar{M}_\theta^{\,2} - 2\nu\, \bar{M}_\xi\, \bar{M}_\theta)/D(1 - \nu^2)]\, \cos\xi\, d\xi \\ - 2\pi a\, [u_o\, \bar{H}(0) + \beta_o\, \bar{M}_\xi(0)] \\ \leq \pi a\, (k_{uH}\, u_o^{\,2} + 2\, k_{uM}\, u_o\, \beta_o + k_{\beta M}\, \beta_o^{\,2})\end{aligned} \tag{166}$$

It is noted that the varied stresses appearing in (166) must satisfy the equilibrium equations and the (stress) boundary conditions at $\xi = \xi_1$. Thus $\bar{N}_\xi$, $\bar{N}_\theta$, $\bar{M}_\xi$ and $\bar{M}_\theta$ must satisfy the following.

a. Equilibrium equations

$$\bar{N}_\xi = - \bar{H} \sin\xi \tag{167}$$

$$\bar{N}_\theta = (\bar{H} \cos\xi)'$$

$$(\bar{M}_\xi \cos\xi)' + \bar{M}_\theta \sin\xi + a\bar{H} \cos^2\xi = 0$$

b. Boundary conditions

$$\bar{H}(\xi_1) = \bar{M}_\xi(\xi_1) = 0 \tag{168}$$

Also, the proof given for the principle of minimum complementary energy required that the varied stresses be continuous.

Let the varied stresses $\bar{H}$ and $\bar{M}_\xi$ be chosen as below.

$$\bar{M}_\xi = [C\ u_o\ g_{1b}(\xi) + aC\ \beta_o\ g_{2b}(\xi)]/\cos\xi \tag{169}$$

$$\bar{H} = [(C/a)\ u_o\ g_{3b}(\xi) + C\ \beta_o\ g_{4b}(\xi)]/\cos\xi \tag{170}$$

The functions g_{1b}, g_{2b}, g_{3b}, g_{4b} are dimensionless and are arbitrary except for the following requirements. Since these requirements are similar to those presented in "Transformation of the Basic Inequality 1," their derivation is not shown.

a. Equilibrium equations

$$\left.\begin{array}{l} g'_{1b} + (\cos\xi)\ g_{3b}(\xi) \\ g'_{2b} + (\cos\xi)\ g_{4b}(\xi) \end{array}\right\} = 0\ (\sin\xi)\ \text{as}\ \xi \to 0 \tag{171}$$

b. Boundary conditions

$$u_o\ g_{1b}(\xi_1) + a\ \beta_o\ g_{2b}(\xi_1) = 0 \tag{172}$$

$$(1/a)\ u_o\ g_{3b}(\xi_1) + \beta_o\ g_{4b}(\xi_1) = 0 \tag{173}$$

c. Continuity: The functions g_{1b}, g_{2b}, g_{3b}, and g_{4b} must be chosen such that $\bar{M}_\xi$ and $\bar{H}$, as defined by (169) and (170), are continuous.

<u>Substitution into the basic inequality 4</u>.--The varied stresses appearing in (166) are expressed in terms of u_o, β_o and g_{1b}, g_{2b}, g_{3b}, g_{4b}. These values are substituted into (166) and a rearrangement of terms yields

$$\begin{aligned}(4D\lambda^3/a^3)\ G_{1b}\ u_o^{\ 2} &+ 2\ (2D\lambda^2/a^2)\ G_{12b}\ u_o\ \beta_o \\ &+ (2D\lambda/a)\ G_{2b}\ \beta_o^{\ 2} \le k_{uH}\ u_o^{\ 2} \\ &+ 2\ k_{uM}\ u_o\ \beta_o + k_{\beta M}\ \beta_o^{\ 2}\ ,\end{aligned} \tag{174}$$

where the quantities F_{1b}, F_{12b}, and F_{2b} are given by expressions similar to equations (141), (142), and (143).

Bounds on the Influence Coefficients

The results of the four previous sections now yield upper and lower bounds for the influence coefficients of the hemispherical shell with apical hole. These bounds are expressed in terms of the corresponding influence coefficients of the semi-infinite circular cylindrical shell. The influence coefficients of the cylindrical shell were obtained from the doctoral dissertation of M. B. Sledd [1] and are shown below. They are distinquished by being enclosed in brackets with the subscript c.

$$[c_{\beta M}]_c = a/D\lambda \tag{175}$$

$$[c_{uM}]_c = a^2/2D\lambda^2$$

$$[c_{uH}]_c = a^3/2D\lambda^3$$

$$[k_{\beta M}]_c = 2D\lambda/a \tag{176}$$

$$[k_{uM}]_c = 2D\lambda^2/a^2$$

$$[k_{uH}]_c = 4D\lambda^3/a^3$$

The transformations of the basic inequalities 1 and 2, (140) and (155), are repeated below with the definitions (175) used.

$$[c_{uH}]_c\, G_{1a}\, H_o^2 - 2\,[c_{uM}]_c\, G_{12a}\, H_o\, M_o \tag{177}$$
$$+ [c_{\beta M}]_c\, G_{2a}\, M_o^2 \geq c_{uH}\, H_o^2$$
$$- 2\, c_{uM}\, H_o\, M_o + c_{\beta M}\, M_o^2$$

$$[c_{uH}]_c\, F_{1a}\, H_o^2 - 2\,[c_{uM}]_c\, F_{12a}\, H_o\, M_o \tag{178}$$
$$+ [c_{\beta M}]_c\, F_{2a}\, M_o^2 \leq c_{uH}\, H_o^2$$
$$- 2\, c_{uM}\, H_o\, M_o + c_{\beta M}\, M_o^2$$

Since the above inequalities hold for arbitrary H_o and M_o, it immediately follows that

$$[c_{uH}]_c\, F_{1a} \leq c_{uH} \leq [c_{uH}]_c\, G_{1a}\ , \tag{179}$$

$$[c_{uM}]_c\, [F_{12a} - \sqrt{(G_{2a} - F_{2a})(G_{1a} - F_{1a})}\ ] \tag{180}$$
$$\leq c_{uM} \leq [c_{uM}]_c\, [G_{12a}$$
$$+ \sqrt{(G_{2a} - F_{2a})(G_{1a} - F_{1a})}\ ]\ ,$$

$$[c_{\beta M}]_c\, F_{2a} \leq c_{\beta M} \leq [c_{\beta M}]_c\, G_{2a}\ . \tag{181}$$

The quantities F_{1a}, F_{12a}, and F_{2a} are given by (156), (157), and (158); the quantities G_{1a}, G_{12a}, and G_{2a} are given by (141), (142), and (143). The functions g_{ia} and f_{ia}, $i = 1, 2, 3, 4$, can now be chosen and the bounds on the direct influence coefficients determined by the evaluation of F_{1a}, F_{12a}, F_{2a}, G_{1a}, G_{12a}, and G_{2a}. The functions g_{ia} and f_{ia}, $i = 1, 2, 3, 4$, must be chosen such that the conditions specified earlier are satisfied. If the bounds thus obtained are to be relatively close, then it is necessary that the varied stresses, (129) and (130), and the varied displacements, (151) and (152), approximate the actual stresses and displacements.

To obtain bounds on the inverse influence coefficients, it is necessary to consider the transformed basic inequalities 3 and 4, (165) and (174). These are repeated below with the notation of equations (176) being used.

$$\begin{aligned} [k_{uH}]_c \, F_{1b} \, u_o^2 + 2 \, [k_{uM}]_c \, F_{12b} \, u_o \, \beta_o \\ + [k_{\beta M}]_c \, F_{2b} \, \beta_o^2 \geq k_{uH} \, u_o^2 \\ + 2 \, k_{uM} \, u_o \, \beta_o + k_{\beta M} \, \beta_o^2 \end{aligned} \tag{182}$$

$$\begin{aligned} [k_{uH}]_c \, G_{1b} \, u_o^2 + 2 \, [k_{uM}]_c \, G_{12b} \, u_o \, \beta_o \\ + [k_{\beta M}]_c \, G_{2b} \, \beta_o^2 \leq k_{uH} \, u_o^2 \\ + 2 \, k_{uM} \, u_o \, \beta_o + k_{\beta M} \, \beta_o^2 \end{aligned} \tag{183}$$

The above inequalities hold for arbitrary u_o and β_o and hence immediately yield the following bounds on the inverse influence coefficients.

$$[k_{uH}]_c \, G_{1b} \leq k_{uH} \leq [k_{uH}]_c \, F_{1b} \tag{184}$$

$$[k_{uM}]_c \, [G_{12b} - \sqrt{(F_{2b} - G_{2b})(F_{1b} - G_{1b})} \,] \leq k_{uM} \leq [k_{uM}]_c \, [F_{12b} + \sqrt{(F_{2b} - G_{2b})(F_{1b} - G_{1b})} \,] \tag{185}$$

$$[k_{\beta M}]_c \, G_{2b} \leq k_{\beta M} \leq [k_{\beta M}]_c \, F_{2b} \tag{186}$$

These bounds can be evaluated by the method suggested for evaluating the bounds on the direct influence coefficients.

CHAPTER V

A NUMERICAL APPLICATION

The techniques previously described are now used to obtain upper and lower bounds on the direct influence coefficients c_{uH} and $c_{\beta M}$ of a particular hemispherical shell with apical hole. Certain physical properties of the shell considered are specified to allow numerical calculations. Thus, the hemispherical shell is specialized by the choice of the following parameters.

$$\xi_1 = \pi/3 \ ,$$

$$\lambda = 12 \ ,$$

$$\nu = 1/4 \ .$$

It is noted that the choice of ξ_1 fixes the size of the apical hole. The quantity λ, (4), is essentially a measure of the ratio of the radius of the middle surface of the shell to the thickness of the shell. The quantity ν is Poisson's ratio of the material comprising the shell.

Evaluation of upper bounds on c_{uH} and $c_{\beta M}$.--Upper bounds on c_{uH} and $c_{\beta M}$ of the hemispherical shell are determined by use of the basic inequality 1, (61).

$$\pi a^2 \int_0^{\xi_1} [(\bar{N}_\xi^2 + \bar{N}_\theta^2 - 2\nu \bar{N}_\xi \bar{N}_\theta)/C + (\bar{M}_\xi^2 + \bar{M}_\theta^2 - 2\nu \bar{M}_\xi \bar{M}_\theta)/D(1-\nu^2)] \cos\xi \, d\xi \geq a\pi(c_{\beta M} M_o^2 - 2 c_{uM} M_o H_o + c_{uH} H_o^2) \ . \tag{187}$$

The varied stresses appearing in (187) must satisfy the equilibrium equations and the stress boundary conditions of Case 1, Chapter III.

Equilibrium equations

$$\bar{N}_\xi = -\bar{H} \sin\xi \tag{188}$$

$$\bar{N}_\theta = (\bar{H} \cos\xi)'$$

$$(\bar{M}_\xi \cos\xi)' + \bar{M}_\theta \sin\xi + a\bar{H} \cos^2\xi = 0$$

Boundary conditions

$$\bar{H}(0) = H_o \ , \ \bar{M}_\xi(0) = M_o \tag{189}$$

$$\bar{H}(\xi_1) = \bar{M}_\xi(\xi_1) = 0$$

Before numerical bounds can be extracted from the basic inequality 1, (187), it is necessary first to choose a form for each of the varied stresses. The varied stresses so chosen must satisfy the above conditions, (188) and (189). The choice of the form of the varied stresses is facilitated by two observations: (1) that a hemispherical shell geometrically approximates a circular cylindrical shell in the vicinity of its equatorial edge and (2) that the known characteristics of the thin, semi-infinite cylindrical shell indicate that forces and moments applied to an edge will produce appreciable deformations only near that edge. Thus it appears reasonable that the expression for the angular displacement β of the hemispherical shell will assume a form similar to the β of the semi-infinite cylindrical shell. Hence, we assume that $\bar{\beta}$ will be of the form

$$\bar{\beta} = e^{\lambda \xi} (c_1 \cos \lambda \xi + c_2 \sin \lambda \xi) + e^{-\lambda \xi} (c_3 \cos \lambda \xi + c_4 \sin \lambda \xi) \tag{190}$$

and thus has the exponential character of the β of the cylindrical shell. The constants c_1, c_2, c_3, and c_4 are to be determined as described shortly.

The basic relationships are consulted for a means of expressing $\bar{M}_\xi$ and $\bar{M}_\theta$ in terms of $\bar{\beta}$. Equations (10), (11), (14), and (15) yield

$$\bar{M}_\xi = (\bar{\beta}' - \nu\bar{\beta}\tan \xi)D/a \ , \tag{191}$$

$$\bar{M}_\theta = (\nu\bar{\beta}' - \bar{\beta}\tan \xi)D/a \ . \tag{192}$$

The third equilibrium equation of (188) provides a means of expressing $\bar{H}$ in terms of $\bar{\beta}$:

$$\bar{H}(\bar{\beta}) = -\left\{ [\bar{M}_{\xi}(\bar{\beta}) \cos\xi]' + \bar{M}_{\theta}(\bar{\beta}) \sin\xi \right\} /(a \cos^2\xi) \tag{193}$$

Since all the equilibrium equations must be satisfied, the remaining equations of (188) are used to determine $\bar{N}_{\xi}$ and $\bar{N}_{\theta}$:

$$\bar{N}_{\xi}(\bar{\beta}) = -\bar{H}(\bar{\beta}) \sin\xi \ , \tag{194}$$

$$\bar{N}_{\theta}(\bar{\beta}) = [\bar{H}(\bar{\beta}) \cos\xi]' \ . \tag{195}$$

Equations (191), (192), (194), and (195) define $\bar{M}_{\xi}$, $\bar{M}_{\theta}$, $\bar{N}_{\xi}$, and $\bar{N}_{\theta}$ in terms of $\bar{\beta}$, (190), in such a way that all of the equilibrium equations are satisfied. The four constants appearing in the expression for $\bar{\beta}$ are so determined that the varied stresses satisfy the four boundary conditions (189).

A set of admissible comparison functions is now available for the integral functional which constitutes the left member of (187). When these expressions are substituted into (187) and the necessary integration performed, the left member assumes the form of a quadratic in H_o and M_o. The reduced state of (187) is shown below.

$$(a/D\lambda)(1.000)\ M_o^{\,2} + A\ M_o\ H_o$$
$$+ (a^3/2\ D\lambda^3)(1.016)\ H_o^{\,2}$$
$$\geq c_{\beta M}\ M_o^{\,2} - 2\ c_{uM}\ M_o\ H_o + c_{uH}\ H_o^{\,2} \tag{196}$$

The quantity A in (196) represents an undetermined constant. Since the inequality (196) holds for arbitrary H_o and M_o, the following inequalities are evident.

$$c_{\beta M} \leq (a/D\lambda)(1.000) \tag{197}$$

$$c_{uH} \leq (a^3/2\ D\lambda^3)(1.016) \tag{198}$$

The notation of (175) is introduced and produces:

$$c_{\beta M} \leq [c_{\beta M}]_c\ (1.000)\ , \tag{199}$$

$$c_{uH} \leq [c_{uH}]_c\ (1.016)\ . \tag{200}$$

Thus the influence coefficient $c_{\beta M}$ of the hemispherical shell with apical hole is no larger than the corresponding influence coefficient of the semi-infinite cylindrical shell, and the influence coefficient c_{uH} of the hemispherical shell is no larger than 1.016 of the corresponding influence coefficient of the cylindrical shell. It is believed that these bounds are correct to the number of significant figures shown.

Evaluation of lower bounds on c_{uH} and $c_{\beta M}$.--Lower bounds on c_{uH} and $c_{\beta M}$ of the hemispherical shell will be evaluated by use of the basic inequality 2, (92):

$$- \pi a^2 \int_0^{\xi_1} [\bar{\epsilon}_{\xi}^2 + \bar{\epsilon}_{\theta}^2 + 2\nu \bar{\epsilon}_{\xi} \bar{\epsilon}_{\theta}) C/(1 - \nu^2) \tag{201}$$

$$+ D(\bar{K}_{\xi}^2 + \bar{K}_{\theta}^2 + 2\nu \bar{K}_{\xi} \bar{K}_{\theta})] \cos \xi \, d\xi$$

$$- 2\pi a \, [\bar{u}(0) \, H_o + \bar{\beta}(0) \, M_o]$$

$$\leq \pi a \, (c_{\beta M} \, M_o^2 - 2 \, c_{uM} \, M_o \, H_o + c_{uH} \, H_o^2) \; .$$

The varied displacements $\bar{u}$ and $\bar{\beta}$ must satisfy only the compatibility equations.

Compatibility equations

$$(\bar{\epsilon}_{\theta} \cos \xi)' + \bar{\epsilon}_{\xi} \sin \xi = \bar{\beta} \cos \xi \tag{202}$$

$$\bar{K}_{\xi} + (\bar{K}_{\theta} \cot \xi)' = 0$$

The varied strains appearing in (202) are defined by the strain-displacement relations:

$$\bar{\epsilon}_{\xi} = (-\bar{u}' + a \bar{\beta} \cos \xi)/(a \sin \xi) \; , \tag{203}$$

$$\bar{\epsilon}_{\theta} = \bar{u}/(a \cos \xi) \; ,$$

$$\bar{K}_{\xi} = \bar{\beta}'/a \; ,$$

$$\bar{K}_{\theta} = (- \bar{\beta} \sin \xi)/(a \cos \xi) \; .$$

For reasons which are similar to those used in the previous section, it is assumed that the displacements of the hemispherical shell can be approximated by the known displacements of the semi-infinite cylindrical shell. Hence, let $\bar{\beta}$ and $\bar{u}$ be chosen as follows.

$$\bar{\beta} = e^{-\lambda\xi} \Big\{ \Big[(a^2/2\ D\lambda^2)\ H_o - (a/D\lambda)\ M_o \Big] \cos\lambda\xi + (a^2/2\ D\lambda^2)\ H_o \sin\lambda\xi \Big\} \tag{204}$$

$$\bar{u} = e^{-\lambda\xi} \Big\{ \Big[(-a^3/2\ D\lambda^3)\ H_o \quad (a^2/2\ D\lambda^2)\ M_o \Big] \cos\lambda\xi - (a^2/2\ D\lambda^2)\ M_o \sin\lambda\xi \Big\} \tag{205}$$

With expressions for $\bar{\beta}$ and $\bar{u}$ available, the varied strains are determined from relations (203) and it is assured that the compatibility equations (202) are automatically satisfied.

The expressions for the varied strains are introduced into the inequality (201) and the necessary integration is performed. The reduced state of (201) is shown below.

$$(a/D\lambda)(0.983)\ M_o^2 + B\ M_o\ H_o \tag{206}$$

$$+ (a^3/2\ D\lambda^3)(0.950)\ H_o^2$$

$$\leq\ c_{\beta M}\ M_o^2 - 2\ c_{uM}\ M_o\ H_o + c_{uH}\ H_o^2$$

The quantity B in (206) represents an undetermined constant. Since the inequality (206) holds for arbitrary H_o and M_o, the following inequalities are evident.

$$(a/D\lambda)(0.983) \leq c_{\beta M} \tag{207}$$

$$(a^3/2\ D\lambda^3)(0.950) \leq c_{uH} \tag{208}$$

The notation of (175) is introduced into (207) and (208) and there results:

$$(0.983)[c_{\beta M}]_c \leq c_{\beta M}\ , \tag{209}$$

$$(0.950)[c_{uH}]_c \leq c_{uH}\ . \tag{210}$$

It is believed that these bounds are correct to the number of significant figures shown.

Upper and lower bounds on c_{uH} and $c_{\beta M}$.--The inequalities (199), (200), (209), and (210) provide upper and lower bounds on the direct influence coefficients c_{uH} and $c_{\beta M}$.

$$(0.983)[c_{\beta M}]_c \leq c_{\beta M} \leq (1.000)[c_{\beta M}]_c \tag{211}$$

$$(0.950)[c_{uH}]_c \leq c_{uH} \leq (1.016)[c_{uH}]_c \tag{212}$$

CHAPTER VI

RESULTS AND CONCLUSIONS

In Chapter III and Chapter IV a technique was presented for obtaining upper and lower bounds on the influence coefficients of the hemispherical shell of constant thickness with apical hole. Bounds of the following form were deduced for each of the influence coefficients.

$$[c_{uH}]_c \, F_{1a} \leq c_{uH} \leq [c_{uH}]_c \, G_{1a} \tag{213}$$

The evaluation of the bounds on the particular influence coefficient c_{uH} involves the computation of the quantities F_{1a} and G_{1a} defined by the relations (156) and (141). These quantities are defined in terms of eight dimensionless functions, $g_{ia}(\xi)$ and $f_{ia}(\xi)$, $i = 1, 2, 3, 4$, that are arbitrary except for certain stated restrictions. Thus, the numerical evaluation of the bounds on c_{uH} has been reduced to a purely mathematical problem. For instance, it is possible that each of the functions g_{ia} and f_{ia} be chosen as a polynomial in ξ with the coefficients initially undetermined. The functions g_{ia} and f_{ia} thus chosen could be made to satisfy the necessary restrictions by proper determination of certain of their coefficients. The quantities F_{1a} and G_{1a} could then be reduced to functions of the still undetermined coefficients of the polynomial representation of f_{1a} and g_{1a}. The methods of the calculus would extremize such functions and numerical bounds on c_{uH} would result. If completely similar techniques were applied to inequalities (180), (181), (184), (185) and (186), bounds on the other influence coefficients would be determined.

Although the above technique for evaluating the bounds on the influence coefficients is mathematically sound, the great amount of labor involved in obtaining good bounds makes the method impractical. Hence, a slightly different approach was used in Chapter V and numerical bounds were obtained on c_{uH} and $c_{\beta M}$ for a particular hemispherical shell with apical hole. These bounds are given by the expressions (211) and (212), and they indicate that $c_{\beta M}$ and c_{uH} for the particular hemispherical shell considered differ by no more than five per cent from the values of the corresponding influence coefficients of the similar semi-infinite cylindrical shell.

Thus the investigation undertaken lends support to the physically reasonable conjecture that the known influence coefficients of the semi-infinite cylindrical shell offer a good approximation of the corresponding influence coefficients of the hemispherical shell of constant thickness with apical hole. The approximation appears to be sufficiently good for many applications.

BIBLIOGRAPHY

Literature Cited

1. Sledd, M. B., On Circular Cylindrical Shells of Variable Wall Thickness, Unpublished Ph. D. Thesis, Massachusetts Institute of Technology, 1954.

Other References

2. Reissner, Eric, "On the Theory of Thin Elastic Shells," Reissner Anniversary Volume: Contributions to Applied Mechanics, J. W. Edwards, Ann Arbor, Mich., 1949, pp. 231-247.

3. Wang, Chi-Teh, Applied Elasticity, New York: McGraw-Hill Book Company, Inc., 1953.

4. Timoshenko, S., Theory of Plates and Shells, New York: McGraw-Hill Book Company, Inc., 1940.

5. Reissner, Eric and Sledd, M. B., Bounds on Influence Coefficients for Circular Cylindrical Shells. AFOSR TN-56-575, ASTIA Document No. 110 397, Report No. 1, Georgia Institute of Technology, 1956.

6. Hildebrand, F. B., "On Asymptotic Integration in Shell Theory," Proceedings of Symposia in Applied Mathematics, Volume III, New York: McGraw-Hill Book Company, Inc., 1950.

7. Gröbner, Wolfgang and Hofreiter, Nikolaus, Integraltafeln, Vienna: Springer-Verlag, 1949.

8. Sokolnikoff, I. S., Mathematical Theory of Elasticity, Second Edition, New York: McGraw-Hill Book Company, Inc., 1956.